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The Multiple Summation Formula and Polylogarithms

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ABSTRACT. In this paper is given the formula:

$$F_n(x) \equiv \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_n=1}^{k_{n-1}} \frac{x^{k_n}}{k_n} = \sum_{\substack{\sum_{j=1}^{n} j\cdot\alpha_j = n \\ \alpha_j \geq 0}} \frac{\prod_{k=1}^{n} \zeta_k^{\alpha_k}(x^k)}{\prod_{k=1}^{n} k^{\alpha_k}\alpha_k!}$$

$$n \geq 1, \quad -1 \leq x < 1$$

with

$$\zeta_k(x) \equiv Li_k(x) \equiv \sum_{r=1}^{\infty} \frac{x^r}{r^k} \quad (k \geq 0),$$

and the method by which it can be obtained.

1. INTRODUCTION

First of all let us introduce the series with theirs notations:

$$\zeta_k(x) \equiv Li_k(x) \equiv \sum_{r=1}^{\infty} \frac{x^r}{r^k} \quad (k \geq 0) \tag{1}$$

It is evident that is $\zeta_0(x) = \frac{x}{1-x}$, $\zeta_1(x) = -\log(1-x)$ $(-1 \leq x < 1)$, and for $k \geq 2$, ζ_k(x) is represented by uniformly and absolutely convergent series in region $-1 \leq x \leq 1$. For these functions, known as polylogarithms and annotated by $Li_k(x)$, an old Legendre's notation will be used in this paper, which also was used by Ramanujan. For functions $F_n(x)$ we can take $F_0(x) = 1$ and for $n = 1$ we have $F_1(x) = \zeta_1(x)$.

Lemma 1.1. *Series for $F_n(x)$ is convergent for $-1 \leq x < 1$ $(n \geq 1)$.*

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Proof. For $|x| \leq |x_0| < 1$ we have:

$$|F_n(x)| \leq F_n(|x|) \leq F_n(|x_0|) < \sum_{k_1=1}^{\infty} \frac{|x_0|^{k_1}}{k_1} \left(\sum_{k_2=1}^{k_1} \frac{1}{k_2} \right)^{n-1} <$$

$$< \sum_{k_1 \geq k_1'} \frac{|x_0|^{k_1}}{k_1} 2^{n-1} \log^{n-1} k_1 + A(k_1') <$$

$$< 2^{n-1} \sum_{k_1 \geq k_1''}^{|x_0|^{k_1}} + A(k_1') + A(k_1'') < +\infty,$$

where $A(k_1')$, $A(k_1'')A(k_1')$, $A(k_1'')$ are positive constants, and:

$$\sum_{k_2=1}^{k_1} \frac{1}{k_2} < 2 \log k_1 \quad (k_1 \geq k_1') \qquad \text{and} \qquad \frac{\log^{n-1} k_1}{k_1} < 1 \quad (k_1 \geq k_1'' \geq k_1').$$

By Weierstrass criterion for uniform convergence we conclude that the power series for $F_n(x)$ is uniformly and absolutely convergent in one closed subdomain of $(-1, 1)$. It has remained to be proved that $F_n(-1)$ exists. In order to prove it let us write:

$$(-1)^{n+1} F_{n+1}(-1) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{k_1}^{k} \frac{(-1)^{k_1-1}}{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} \frac{(-1)^{k_n-1}}{k_n} =$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \upsilon(n, k) = \sum_{k=1}^{\infty} (-1)^{k-1} a_k.$$

If we **prove**:

1. $\lim\limits_{k \to \infty} a_k = 0$,
2. $d_t^{(s)} = \frac{1}{2t-1} \upsilon(s, 2t-1) - \frac{1}{2t} \upsilon(s, 2t) > 0 \quad \Rightarrow \quad a_{2t-1} > a_{2t} \quad (\forall t \in N)$,
3. $\upsilon(s, t) > 0 \quad (s, t \in N) \quad \Rightarrow \quad a_k > 0 \quad (\forall k \geq 1)$,

from Leibnitz criterion for alternatively series will follow the convergence of series:

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

and the existence of $F_n(-1)$.

Indeed, we have:

1.

$$0 \le \left| \lim_{k\to\infty} a_k \right| = \lim_{k\to\infty} \frac{|\upsilon(n,k)|}{k} \le \lim_{k\to\infty} \frac{1}{k} \left(\sum_{k_1=1}^{k} \frac{1}{k_1} \right)^{n-1} \le$$

$$\le 2^{n-1} \lim_{k\to\infty} \frac{\log^{n-1} k}{k} = 0 \quad \Rightarrow$$

$$\lim_{k\to\infty} a_k = 0.$$

2. Prove by induction for s.

$$d_t^{(1)} = \frac{1}{(2t-1)2t} \left\{ \sum_{j=1}^{2t} \frac{(-1)^{j-1}}{j} + 1 \right\} > 0 \qquad (\forall t),$$

$$d_t^{(2)} = \frac{1}{(2t-1)2t} \{\upsilon(2,2t) + \upsilon(1,2t-1)\},$$

$$\upsilon(2,2t) = \sum_{j=1}^{t} d_j^{(1)} > 0, \qquad \upsilon(1,2t-1) > 0 \quad \Rightarrow \quad d_t^{(2)} > 0 \quad (\forall t),$$

$$d_t^{(3)} = \frac{1}{(2t-1)2t} \{\upsilon(3,2t) + \upsilon(2,2t-1)\},$$

$$\left.\begin{aligned} \upsilon(3,2t) &= \textstyle\sum_{j=1}^{t} d_j^{(2)} > 0, \\ \upsilon(2,2t-1) &= \upsilon(2,2t) + \tfrac{1}{2t}\upsilon(1,2t) > 0 \end{aligned}\right\} \quad \Rightarrow \quad d_t^{(3)} > 0 \quad (\forall t),$$

$$d_t^{(s+1)} = \frac{1}{(2t-1)2t} \{\upsilon(s+1,2t) + \upsilon(s,2t-1)\},$$

$$\upsilon(s+1,2t) = \sum_{j=1}^{t} d_j^{(s)} > 0, \quad \upsilon(s,2t-1) = \upsilon(s,2t) + \frac{1}{2t}\upsilon(s-1,2t)$$

$$\Rightarrow \quad d_t^{(s+1)} = \frac{1}{(2t-1)2t} \left\{ \sum_{j=1}^{t} d_j^{(s)} + \sum_{j=1}^{t} d_j^{(s-1)} + \frac{1}{2t} \sum_{j=1}^{t} d_j^{(s-2)} \right\} \quad (s \ge 3, \forall t).$$

By assumption of induction, from here, we conclude that is $d_t^{(s+1)} > 0$ and by induction 2. is proved.

3.

$$\upsilon(1,t) > 0 \quad (\forall t),$$

$$\upsilon(2,t) > 0 \quad (\forall t) \qquad \text{(follows from 2.)},$$

$$\upsilon(s,2t) = \sum_{j=1}^{t} d_j^{(s-1)} > 0 \quad (s \geq 2) \qquad \text{(follows from 2.)},$$

$$\upsilon(s,2t-1) = \upsilon(s,2t) + \frac{1}{2t}\upsilon(s-1,2t) > 0 \quad (s \geq 2).$$

Thus, we have proved lemma 1. □

Lemma 1.2.

(2) $$F_r'(x) = \frac{1}{x}\sum_{j=0}^{r-1}\zeta_j(x^{j+1})F_{r-1-j}(x) \qquad (-1 \leq x < 1),\ (r \geq 1).$$

Proof.

$$\begin{aligned}
F_r'(x) &= \sum_{k_1=1}^{\infty} x^{k_1-1}\sum_{k_2=1}^{k_1}\frac{x^{k_2}}{k_2}\cdots\sum_{k_r=1}^{k_{r-1}}\frac{x^{k_r}}{k_r}+\\
&+\sum_{p=2}^{r-1}\sum_{k_1=1}^{\infty}\frac{x^{k_1}}{k_1}\cdots\sum_{k_p=1}^{k_{p-1}}x^{k_p-1}\cdots\sum_{k_r=1}^{k_{r-1}}\frac{x^{k_r}}{k_r}+\\
&+\sum_{k_1=1}^{\infty}\frac{x^{k_1}}{k_1}\sum_{k_2=1}^{k_1}\frac{x^{k_2}}{k_2}\cdots\sum_{k_{r-1}=1}^{k_{r-2}}\frac{x^{k_{r-1}}}{k_{r-1}}\sum_{k_r=1}^{k_{r-1}}\frac{x^{k_r}}{k_r}=\\
&= S_1 + \sum_{p=2}^{r-1} S_p + S_r;
\end{aligned}$$

$$\begin{aligned}
S_1 &= \sum_{k_2=1}^{\infty}\frac{x^{k_2}}{k_2}\cdots\sum_{k_r=1}^{k_{r-1}}\frac{x^{k_r}}{k_r}\sum_{k_1=k_2}^{\infty}x^{k_1-1}=\\
&= \frac{1}{x(1-x)}\sum_{k_2=1}^{\infty}\frac{x^{2k_2}}{k_2}\sum_{k_3=1}^{k_2}\frac{x^{k_3}}{k_3}\cdots\sum_{k_r=1}^{k_{r-1}}\frac{x^{k_r}}{k_r}=\\
&= \frac{1}{x(1-x)}\sum_{k_1=1}^{\infty}\frac{x^{2k_1}}{k_1}\sum_{k_2=1}^{k_1}\frac{x^{k_2}}{k_2}\cdots\sum_{k_{r-1}=1}^{k_{r-2}}\frac{x^{k_{r-1}}}{k_{r-1}}=\\
&= \frac{1}{x(1-x)}T_1.
\end{aligned}$$

Let T_p be the series obtain from $F_{r-1}(x)$ substituting x^{k_p} by x^{2k_p} $\quad (1 \le p \le r)$.

$$S_p = \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \cdots \sum_{k_{p-1}=1}^{k_{p-2}} \frac{x^{k_{p-1}}}{k\quad_{p-1}} \sum_{k_{p+1}=1}^{k_{p-1}} \frac{x^{k_{p+1}}}{k_{p+1}} \cdots \sum_{k_r=1}^{k_{r-1}} \frac{x^{k_r}}{k_r} \sum_{k_p=k_{p+1}}^{k_{p-1}} x^{k_p-1} =$$

$$= \frac{1}{x(1-x)} T_p - \frac{1}{1-x} T_{p-1} \qquad (2 \le p \le r-1).$$

In the same way it follows that:

$$S_r = \frac{1}{1-x} F_{r-1}(x) - \frac{1}{1-x} T_{r-1} \quad \Rightarrow$$

$$\Rightarrow \quad F_r'(x) = \frac{1}{x(1-x)} T_1 + $$

$$+ \sum_{p=2}^{r-1} \left\{ \frac{1}{x(1-x)} T_p - \frac{1}{1-x} T_{p-1} \right\} +$$

$$+ \frac{1}{1-x} F_{r-1}(x) - \frac{1}{1-x} T_{r-1} \quad \Rightarrow$$

$$\Rightarrow \quad F_r'(x) = \frac{1}{x} \sum_{p=1}^{r-1} T_p + \frac{1}{1-x} F_{r-1}(x). \tag{3}$$

Further,

$$T_{r-1} = \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{k_{r-2}}}{k_{r-2}} \sum_{k_{r-1}=1}^{k_{r-2}} \frac{x^{2k_{r-1}}}{k_{r-1}} =$$

$$= \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-1}=1}^{k_{r-3}} \frac{x^{2k_{r-1}}}{k_{r-1}} \sum_{k_{r-2}=k_{r-1}}^{k_{r-3}} \frac{x^{k_{r-2}}}{k_{r-2}} =$$

$$= \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{3k_{r-2}}}{k_{r-2}^2} +$$

$$+ \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{2k_{r-2}}}{k_{r-2}} \sum_{k_{r-1}=1}^{k_{r-3}} \frac{x^{k_{r-1}}}{k_{r-1}} - T_{r-2} =$$

$$= \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-2}=1}^{k_{r-4}} \frac{x^{3k_{r-2}}}{k_{r-2}^2} \sum_{k_{r-3}=k_{r-2}}^{k_{r-4}} \frac{x^{k_{r-3}}}{k_{r-3}} +$$

$$+ \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-2}=1}^{k_{r-4}} \frac{x^{2k_{r-2}}}{k_{r-2}} \sum_{k_{r-3}=k_{r-2}}^{k_{r-4}} \frac{x^{k_{r-3}}}{k_{r-3}} \sum_{k_{r-1}=1}^{k_{r-3}} \frac{x^{k_{r-1}}}{k_{r-1}} - T_{r-2} =$$

$$= \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-3}=1}^{k_{r-4}} \frac{x^{4k_{r-3}}}{k_{r-3}^3} +$$

$$+ \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-3}=1}^{k_{r-4}} \frac{x^{3k_{r-3}}}{k_{r-3}^2} \sum_{k_{r-2}=1}^{k_{r-4}} \frac{x^{k_{r-2}}}{k_{r-2}} +$$

$$+ \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-3}=1}^{k_{r-4}} \frac{x^{2k_{r-3}}}{k_{r-3}} \sum_{k_{r-2}=1}^{k_{r-4}} \frac{x^{k_{r-2}}}{k_{r-2}} \sum_{k_{r-1}=1}^{k_{r-2}} \frac{x^{k_{r-1}}}{k_{r-1}} - T_{r-3} - T_{r-2} =$$

$$= \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-4}=1}^{k_{r-5}} \frac{x^{5k_{r-4}}}{k_{r-4}^4} +$$

$$+ \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-4}=1}^{k_{r-5}} \frac{x^{4k_{r-4}}}{k_{r-4}^3} \sum_{k_{r-3}=1}^{k_{r-5}} \frac{x^{k_{r-3}}}{k_{r-3}} +$$

$$+ \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-4}=1}^{k_{r-5}} \frac{x^{3k_{r-4}}}{k_{r-4}^2} \sum_{k_{r-3}=1}^{k_{r-5}} \frac{x^{k_{r-3}}}{k_{r-3}} \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{k_{r-2}}}{k_{r-2}} +$$

$$+ \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{3k_{r-2}}}{k_{r-2}^2} +$$

$$+ \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{2k_{r-2}}}{k_{r-2}} \sum_{k_{r-1}=1}^{k_{r-3}} \frac{x^{k_{r-1}}}{k_{r-1}} - T_{r-2} =$$

$$= \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_{r-4}=1}^{k_{r-5}} \frac{x^{2k_{r-4}}}{k_{r-4}} \sum_{k_{r-3}=1}^{k_{r-5}} \frac{x^{k_{r-3}}}{k_{r-3}} \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{k_{r-2}}}{k_{r-2}} \sum_{k_{r-1}=1}^{k_{r-2}} \frac{x^{k_{r-1}}}{k_{r-1}} -$$

$$- T_{r-4} - T_{r-3} - T_{r-2} = \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{2k_2}}{k_2} \sum_{k_3=1}^{k_1} \frac{x^{k_3}}{k_3} \cdots \sum_{k_{r-1}=1}^{k_{r-2}} \frac{x^{k_{r-1}}}{k_{r-1}} +$$

$$+ \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{3k_2}}{k_2^2} \sum_{k_3=1}^{k_1} \frac{x^{k_3}}{k_3} \cdots \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{k_{r-2}}}{k_{r-2}} + \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{4k_2}}{k_2^3} \sum_{k_3=1}^{k_1} \frac{x^{k_3}}{k_3} \cdots$$

$$\cdots \sum_{k_{r-3}=1}^{k_{r-4}} \frac{x^{k_{r-3}}}{k_{r-3}} + \cdots + \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{(r-1)k_2}}{k_2^{r-2}} - \sum_{j=2}^{r-2} T_j =$$

$$-\sum_{j=2}^{r-2} T_j + \left(\sum_{k_2=1}^{\infty} \frac{x^{3k_2}}{k_2^2} \sum_{k_3=1}^{k_2} \frac{x^{k_3}}{k_3} \cdots \sum_{k_{r-1}=1}^{k_{r-2}} \frac{x^{k_{r-1}}}{k_{r-1}} + \zeta_1(x^2) F_{r-2}(x) - T_1\right) +$$

$$+ \left(\sum_{k_2=1}^{\infty} \frac{x^{4k_2}}{k_2^3} \sum_{k_3=1}^{k_2} \frac{x^{k_3}}{k_3} \cdots \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{k_{r-2}}}{k_{r-2}} + \zeta_2(x^3) F_{r-3}(x) - \right.$$

$$\left. - \sum_{k_2=1}^{\infty} \frac{x^{3k_2}}{k_2^2} \sum_{k_3=1}^{k_2} \frac{x^{k_3}}{k_3} \cdots \sum_{k_{r-1}=1}^{k_{r-2}} \frac{x^{k_{r-1}}}{k_{r-1}}\right) + \left(\sum_{k_2=1}^{\infty} \frac{x^{5k_2}}{k_2^4} \sum_{k_3=1}^{k_2} \frac{x^{k_3}}{k_3} \cdots \right.$$

$$\left. \cdots \sum_{k_{r-3}=1}^{k_{r-4}} \frac{x^{k_{r-3}}}{k_{r-3}} + \zeta_3(x^4) F_{r-4}(x) - \sum_{k_2=1}^{\infty} \frac{x^{4k_2}}{k_2^3} \sum_{k_3=1}^{k_2} \frac{x^{k_3}}{k_3} \cdots \sum_{k_{r-2}=1}^{k_{r-3}} \frac{x^{k_{r-2}}}{k_{r-2}}\right) + \cdots +$$

$$+ (\zeta_{r-1}(x^r) + \zeta_{r-2}(x^{r-1}) F_1(x) - \sum_{k_2=1}^{\infty} \frac{x^{(r-1)k_2}}{k_2^{r-2}} \sum_{k_3=1}^{k_2} \frac{x^{k_3}}{k_3}) =$$

$$= -\sum_{j=1}^{r-2} T_j + \sum_{j=1}^{r-1} \zeta_j(x^{j+1}) F_{r-1-j}(x) \quad \Rightarrow$$

(4)
$$T_{r-1} = -\sum_{j=1}^{r-2} T_j + \sum_{j=1}^{r-1} \zeta_j(x^{j+1}) F_{r-1-j}(x).$$

From (3) and (4) with $\zeta_0(x) = \frac{x}{1-x}$ follows formula (2). □

Lemma 1.3.

(5)
$$nF_n(x) = \sum_{j=0}^{n-1} \zeta_{n-j}(x^{n-j}) F_j(x) \qquad (-1 \le x < 1) \ \ (n \ge 1).$$

Proof. Formula (5) can be proved by induction. For $n = 1$ it is true because of $F_0(x) = 1, \quad F_1(x) = \zeta_1(x)$.

(6)
$$\left(\sum_{j=0}^{n} \zeta_{n+1-j}(x^{n+1-j}) F_j(x)\right)' = \frac{1}{x} \sum_{j=0}^{n} (n+1-j) \zeta_{n-j}(x^{n+1-j}) F_j(x) + $$
$$+ \sum_{j=0}^{n} \zeta_{n+1-j}(x^{n+1-j}) F_j'(x),$$

since

(7)
$$(\zeta_k(x^k))_x' = \frac{k}{x} \zeta_{k-1}(x^k).$$

Second sum on the right-hand side of (6) is:

$$\sum_{j=0}^{n}\zeta_{n+1-j}(x^{n+1-j})F_j'(x)\overset{(2)}{=} \tag{8}$$

$$=\frac{1}{x}\sum_{j=1}^{n}\zeta_{n+1-j}(x^{n+1-j})\sum_{s=0}^{j-1}\zeta_s(x^{s+1})F_{j-1-s}(x)=$$

$$=\frac{1}{x}\sum_{s=0}^{n-1}\zeta_s(x^{s+1})\sum_{j=s+1}^{n}\zeta_{n+1-j}(x^{n+1-j})F_{j-1-s}(x)=$$

$$=\frac{1}{x}\sum_{s=0}^{n-1}\zeta_s(x^{s+1})\sum_{t=0}^{(n-s)-1}\zeta_{(n-s)-t}(x^{(n-s)-t})F_t(x)\overset{(*)}{=}$$

$$=\frac{1}{x}\sum_{s=0}^{n-1}(n-s)\zeta_s(x^{s+1})F_{n-s}(x)=\frac{1}{x}\sum_{j=0}^{n}j\zeta_{n-j}(x^{n+1-j})F_j(x).$$

Sign (*) denotes that we applied assumption of induction that formula (5) is true for every $1\le k\le n$.

From (6) and (8) we get:

$$\left(\sum_{j=0}^{n}\zeta_{n+1-j}(x^{n+1-j})F_j(x)\right)'=\frac{n+1}{x}\sum_{j=0}^{n}\zeta_{n-j}(x^{n+1-j})F_j(x)\overset{(2)}{=}$$

$$=((n+1)F_{n+1}(x))' \quad\Rightarrow$$

$$\Rightarrow\quad (n+1)F_{n+1}(x)=\sum_{j=0}^{n}\zeta_{n+1-j}(x^{n+1-j})F_j(x)+C.$$

and for $x=0$ we have $C=0$, so the formula (5) is true for $n+1$, and by induction it is true for every natural n. □

Theorem 1.1.

$$F_n(x)\equiv\sum_{k_1=1}^{\infty}\frac{x^{k_1}}{k_1}\sum_{k_2=1}^{k_1}\frac{x^{k_2}}{k_2}\cdots\sum_{k_n=1}^{k_{n-1}}\frac{x^{k_n}}{k_n}=\sum_{\substack{\sum_{j=1}^{n}j\cdot\alpha_j=n\\ \alpha_j\ge 0}}\frac{\prod_{k=1}^{n}\zeta_k^{\alpha_k}(x^k)}{\prod_{k=1}^{n}(k^{\alpha_k}\alpha_k!)} \tag{9}$$

$$(n\ge 1),\quad (-1\le x<1).$$

Proof. Formula (9) follows from (5) and from assumption of induction that formula (9) is true for every $1\le k\le n-1$. In sum $\frac{1}{n}\sum_{j=1}^{n}\zeta_j(x^j)F_{n-j}(x)$, for some combination $\sum_{j=1}^{n}j(\alpha_j)_0=n$, in numerators for $j=1,2,\ldots,n$ appear respectively (some of them is obtained from the previous ones) $j(\alpha_j)_0$, which in sum gives:

$$\frac{1}{n}\sum_{j=1}^{n} j(\alpha_j)_0 \frac{\prod_{k=1}^{n} \zeta_k^{(\alpha_k)_0}(x^k)}{\prod_{k=1}^{n} (k^{(\alpha_k)_0}(\alpha_k)_0!)},$$

which at the end leads to the formula (9) for n. □

Corollary 1.1. *Using the formulas:*

$$\zeta_k(-1) = -\left(1 - \frac{1}{2^{k-1}}\right)\zeta_k(1) \quad (k \geq 2), \quad \zeta_1(-1) = -\log 2,$$

with $\zeta_k = \zeta(k)$, $\zeta_k(-1) = \zeta(-k)$, *from (9) we get:*

$$F_3(-1) = -\frac{1}{6}\log^3 2 - \frac{1}{2}\zeta(2)\log 2 - \frac{1}{4}\zeta(3),$$

$$F_4(-1) = -\frac{1}{24}\log^4 2 + \frac{1}{4}\zeta(2)\log^2 2 + \frac{1}{4}\zeta(3)\log 2 + \frac{9}{16}\zeta(4),$$

$$\begin{aligned} F_5(-1) = & -\frac{1}{120}\log^5 2 - \frac{1}{12}\zeta(2)\log^3 2 - \frac{1}{8}\zeta(3)\log^2 2 - \\ & -\frac{9}{16}\zeta(4)\log 2 - \frac{1}{8}\zeta(3)\zeta(2) - \frac{3}{16}\zeta(5). \end{aligned}$$

Corollary 1.2. *From recurrence relation (5) it follows:*

$$\exp\left(\sum_{j=1}^{\infty} \frac{\zeta_j(x^j)}{j} t^j\right) = \sum_{n=0}^{\infty} F_n(x) t^n \qquad (-1 \leq x < 1). \tag{10}$$

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